Linear maps

Two and a half weeks

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Module

This chapter is part of the ALM module: linear maps and matrices. Its is part of the B4 bimester.

The prerequisites are:

- Module [Maths-S1B1 OBM] from the seminar.
- Module [Maths-S2B1-B2 EV] about vector spaces.

Context

Linear maps are underlying in many mathematical situations, such as some recursive sequences or *linear* differential equations that you have studied before. When we define a numerical sequence by $u_0 = 1$ and

$$\forall n \in \mathbb{N}, \quad u_{n+1} = 2u_n$$

we require that u_{n+1} be the image of the term u_n by the linear map $f : x \mapsto 2x$. A similar situation occurs when solving the differential equation

$$f' = 2f$$

These examples can be extended to higher dimensions, such as

$$\forall n \in \mathbb{N},\ u_{n+2} = 2u_{n+1} - u_n \qquad \text{or} \qquad f'' = 2f' - f$$

They motivate our focus on the linear maps. However, another motivation may be stronger: modern maths emphasize not so much the objects for themselves, but the operators which connect and transform the objects. These operators are mappings, they map an object to another one or to its transformed version. For instance, it is more important to know how you can change your GPS position, than to know which position you had at your day of birth.

If we think in terms of Object Oriented Programming, we can say that a class is only a data set if it has no methods. Linear maps are the *methods* of the abstract class ***vector***.

Learning outcomes

At the end of this chapter, you are expected to be able to:

- Recognize a linear map (for example a projector or a symmetry).
- Find the kernel and the image of a linear map (for example, using the rank-nullity theorem).

More precisely, you should be able to:

- Prove whether a mapping is linear or not.
- Find the kernel and the image of a linear map, and find a basis when they are finite-dimensional.
- Use the rank-nullity theorem to validate or anticipate the kernel/image dimensions, when the input space is finite-dimensional.
- Prove whether or not a linear map is injective, surjective, bijective.
- Analyze a projector.
- Represent graphically the action of a projector or a symmetry.

1 Definition of a linear map

1.1 Summary

The purpose of this section is to feel what a linear map is. A mapping is a process which relates an element from an input set to an element from an output set. If both sets are vector spaces, the linearity is the property that the mapping is consistent with the vector space structure of the sets.

Formally, a mapping f from a vector space E to a vector space F is linear if for any $(u, v) \in E^2$ and any $\lambda \in \mathbb{R}$,

$$f(u+v) = f(u) + f(v) \qquad \text{and} \qquad f(\lambda.u) = \lambda.f(u)$$

It is hence consistent with the operators $+$ and $.$ of the spaces E and F.

1.2 Available MiMos

Mimo: Définition d'une application linéaire

Mimo: Exemples d'applications linéaires

1.3 Exercises

Exercise 1.1

To start with, we study numerical functions with one real variable. They are the mappings that you have handled the most.

1. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. What do you understand if we say that f is linear? How is its graph?
2. Let u and v be two real numbers, what is $f(u+v)$? And for $\alpha \in \mathbb{R}$, what is $f(\alpha u)$? What is the value of $f(0)$?
3. Same questions for an affine function g from $\mathbb{R}$ to $\mathbb{R}$.
4. Answer to the previous questions for the function $h : x \longmapsto x^2$.

Exercise 1.2

Are the following mappings linear?

$$f_1 : \begin{cases} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ (x,y,z) & \longmapsto & (2x-z, 3x+y+z) \end{cases}$$

$$f_2 : \begin{cases} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ (x,y,z) & \longmapsto & (2x-z, 3x+2+z) \end{cases}$$

$$f_3 : \begin{cases} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ (x,y) & \longmapsto & (x+y, -y, 2x-y) \end{cases}$$

$$f_4 : \begin{cases} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ (x,y) & \longmapsto & (x-y, xy, 2x) \end{cases}$$

$\star$ $f_5 : \left\{ \begin{array}{ccc} \mathbb{R}^p & \longrightarrow & \mathbb{R}^n \\ \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} & \longmapsto & \begin{pmatrix} a_{11}x_1 + \cdots + a_{1p}x_p \\ \vdots \\ a_{n1}x_1 + \cdots + a_{np}x_p \end{pmatrix} \end{array} \right.$ where the coefficients a_{ij} are real numbers.

f_6 : **(Personal work)** $\left\{ \begin{array}{ccc} E & \longrightarrow & E \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \begin{pmatrix} a-b & b-c \\ d & d \end{pmatrix} \end{array} \right.$

The vector space $E = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (a,b,c,d) \in \mathbb{R}^4 \right\}$ has been studied in the previous worksheet.

$f_7 : \left\{ \begin{array}{ccc} \mathbb{R}^{\mathbb{N}} & \longrightarrow & \mathbb{R}^2 \\ u & \longmapsto & (u_0, u_1) \end{array} \right.$.

$f_8 : \left\{ \begin{array}{ccc} \mathbb{R}^{\mathbb{N}} & \longrightarrow & \mathbb{R}^2 \\ u & \longmapsto & (u_0, \sqrt{u_1}) \end{array} \right.$.

$f_9 : \left\{ \begin{array}{ccc} \mathbb{R}[X] & \longrightarrow & \mathbb{R}[X] \\ P & \longmapsto & 2(X+1)P - X^2 \end{array} \right.$.

$f_{10} : \left\{ \begin{array}{ccc} \mathbb{R}[X] & \longrightarrow & \mathbb{R}[X] \\ P & \longmapsto & 2(X+1)P - (X^2 - 2X + 1)P' \end{array} \right.$.

$\phi : \left\{ \begin{array}{ccc} \mathscr{C}^0(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ f & \longmapsto & \displaystyle\int_0^1 f(t)\mathrm{d}t \end{array} \right.$ where $\mathscr{C}^0(\mathbb{R})$ is the set of all the continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

2 Kernel and image of a linear map

2.1 Summary

Studying a linear map involves studying two linear subspaces associated to the linear map: its ***kernel*** and its ***image***. The first one is the area in the input space which leaves (nearly) no trail in the output space, while the second one is the area in the output space where the trail is. Thus, the kernel is in the input set, the image in the output set.

2.2 Available MiMos

Mimo: Image et noyau d'une application linéaire

2.3 Exercises

Exercise 2.3

1. Let E and F be two $\mathbb{R}$-vector spaces and $f \in \mathcal{L}(E, F)$. Remind the definitions of the kernel of f and the image of f. Show that the kernel of f is a linear subspace of E and the image of f a linear subspace of F.
2. Let $f : \begin{cases} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & (x + y, 0) \end{cases}$
   (a) Show that f is a linear map.
   (b) Find the kernel of f and draw it on the plane.
   (c) Find the image of f and draw it on the plane.
   (d) What can be said about these subsets of $\mathbb{R}^2$?

Exercise 2.4

Find the kernel and the image of the following linear maps. Express them as spanned linear subspaces (using the Span notation) when it is possible.

$$f_1 : \begin{cases} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ (x, y, z) & \longmapsto & (x - y, 2x - y + z) \end{cases}$$

$$f_2 : \begin{cases} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (x, y, z) & \longmapsto & (x, x + z, z) \end{cases}$$

$$f_3 : \begin{cases} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ (x, y) & \longmapsto & (x, 2x + y, 3y) \end{cases}$$

$$f_4 : \begin{cases} \mathbb{R}_2[X] & \longrightarrow & \mathbb{R}_2[X] \\ P & \longmapsto & P' \end{cases}$$

Same question if $f_4 : \mathbb{R}[X] \longrightarrow \mathbb{R}[X]$.

$$f_5 : \begin{cases} \mathbb{R}[X] & \longrightarrow & \mathbb{R}^2 \\ P & \longmapsto & (P(0), P(1)) \end{cases}$$

Exercise 2.5

Let E and F be two $\mathbb{R}$-vs and $f \in \mathscr{L}(E, F)$.

1. (a) Show that if f is injective then: $\forall u \in E,\ f(u) = 0_F \Longrightarrow u = 0_E$
   (b) Show that if $\mathrm{Ker}(f) = \{0_E\}$ then: $\forall (u, v) \in E^2,\ f(u) = f(v) \Longrightarrow u = v$
   (c) Show that f is injective iff $\mathrm{Ker}(f) = \{0_E\}$.
2. Show that f is surjective iff $\mathrm{Im}(f) = F$.

Exercise 2.6

For each of the linear maps from exercise 2-4, say whether it is injective and/or surjective.

3 Rank-nullity theorem

3.1 Summary

The sizes of the kernel and the image of a linear map are related. You may have noticed, in the examples above, that a linear map with a *large* kernel has a *small* image and that, reversely, a linear map with a *small* kernel has a *large* image. This intuition is true when the involved vector spaces are finite-dimensional. It is formalized by the ***rank-nullity theorem***, which provides an explicit relation between the ***dimensions*** of the kernel and the image. Note that this intuition is not true when the spaces are not finite-dimensional.

The rank-nullity theorem is very convenient in many situations. It enables one to use dimension arguments when proving properties, which can strongly simplify the proofs.

Rank-nullity theorem:
Let E and F be two $\mathbb{R}$-vs, such that E is finite-dimensional.
Let $f \in \mathscr{L}(E, F)$. Then:

- $\mathrm{Im}(f)$ is finite-dimensional. Its dimension is the rank of f and is denoted by $\mathrm{rank}(f)$.
- The following relation holds: $\dim \mathrm{Im}(f) + \dim \mathrm{Ker}(f) = \dim E$

This section highlights this property and shows typical situations where it helps understanding a linear map.

3.2 Exercises

Exercise 3.7
For each linear map from exercise 2-4, find the dimensions of its kernel and its image. Then check that the rank-nullity theorem is satisfied.

Exercise 3.8
Consider the linear map $f : \begin{cases} \mathbb{R}_2[X] & \longrightarrow & \mathbb{R}^2 \\ P = aX^2 + bX + c & \longmapsto & (a + c, a + 2b + 3c) \end{cases}$

1. Find the dimension of $\mathrm{Ker}(f)$. Justify accurately your reasoning.
2. What is the dimension of $\mathrm{Im}(f)$?
3. Find $\mathrm{Im}(f)$.
4. Is f injective? surjective? bijective?

Exercise 3.9
(Personal work) Let $\mathscr{B} = (e_1, e_2, e_3)$ be the standard basis of $\mathbb{R}^3$ and consider the linear map

$$f : \begin{cases} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (x, y, z) & \longmapsto & (6x - 4y - 4z, 5x - 3y - 4z, x - y) \end{cases}$$

1. Find a basis of $\mathrm{Ker}(f)$ and determine its dimension.
2. Deduce the dimension of $\mathrm{Im}(f)$.
3. Let $u = e_1 + e_2$ and $v = e_2 - e_3$. Compute $f(u)$ and $f(v)$.
4. Deduce that (u, v) is a basis of $\mathrm{Im}(f)$.

Exercise 3.10

1. Let E be a finite-dimensional vector space and $f \in \mathscr{L}(E)$ an endomorphism of E.
   Using the rank-nullity theorem, show that:
   (a) if f is injective then f is surjective,
   (b) if f is surjective then f is injective,
   (c) f bijective $\iff \mathrm{Ker}(f) = \{0_E\} \iff \mathrm{Im}(f) = E$.
2. Let F be a second finite-dimensional vector space and $g \in \mathscr{L}(E, F)$.
   What can we say about g according to the dimensions of E and F?

★ **Exercice 3.11**

Let $(x_0, x_1, \cdots, x_n)$ be a $(n+1)$-tuple of distinct real numbers and f the linear map defined by:

$$f : \begin{cases} \mathbb{R}_n[X] & \longrightarrow & \mathbb{R}^{n+1} \\ P & \longmapsto & (P(x_0), P(x_1), \cdots, P(x_n)) \end{cases}$$

1. Show that f is injective, then that it is bijective.
2. Deduce that, for any set of $n+1$ points in the plane with distinct x values, there exists a unique polynomial of degree n which fits these points.

Exercise 3.12

(Personal work) Let $E = \mathbb{R}[X]$ and $f : \begin{cases} \mathbb{R}[X] & \longrightarrow & \mathbb{R}[X] \\ P & \longmapsto & P' \end{cases}$.

1. Show that f is an endomorphism of E.
2. Determine its kernel and its image.
3. Is f injective? surjective?
4. What can you deduce from this result?

4 Projectors and symmetries

4.1 Summary

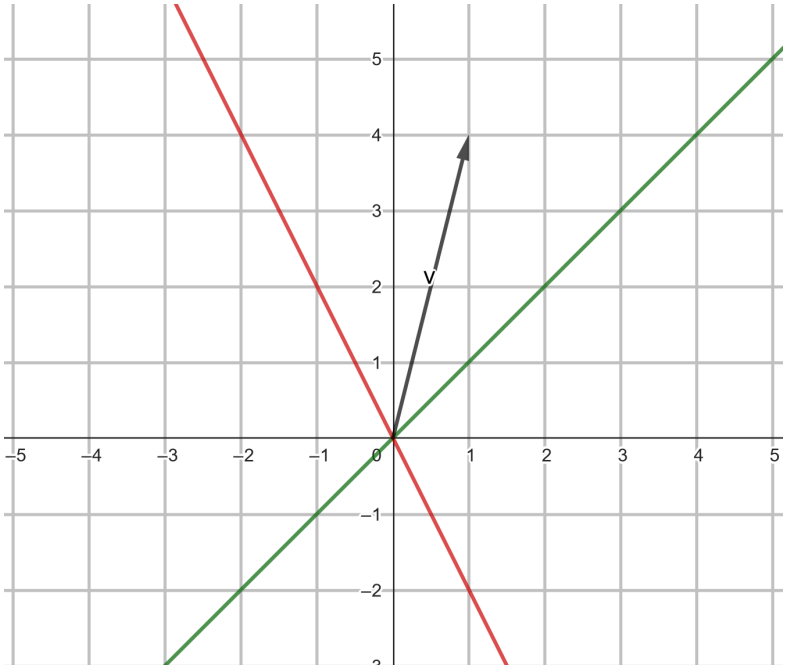
Some of the linear maps are more frequent in real life situations: they are the projectors. For example, they enable one to project some parts of $\mathbb{R}^3$ onto a plane, as required when representing the earth on a map. These operators are also used when we need to reduce the dimension of an image (data compression) without losing their *useful information*, the one that our eyes will see.

This section shows examples. It highlights some of the projectors' properties, and those of the closely related operators, the symmetries.

4.2 Exercises

Exercise 4.13

The figure below represents in $\mathbb{R}^2$ the line L_1 of equation $y = x$, the line L_2 of equation $y = -2x$ and the vector $v = (1, 4)$.



1. In term of vector spaces theory, what can you say about the lines L_1 and L_2?
2. Build graphically the projection v_1 of v on L_1 parallel to L_2, then the projection v_2 of v on L_2 parallel to L_1. What is the relation between v, v_1 and v_2?
3. Find graphically the coordinates of v_1 and v_2. Check that the relation of previous question is satisfied.
4. What is the projection of v_1 on L_1 parallel to L_2? What is the projection of v_2 on L_1 parallel to L_2?
5. Build graphically the symmetry of v about L_1 parallel to L_2.

Exercise 4.14

Let p_1, p_2 be two endomorphisms of $\mathbb{R}^2$ defined by:

$$p_1 : \begin{cases} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & (x, 0) \end{cases}$$

$$p_2 : \begin{cases} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & (x - y, 0) \end{cases}$$

For each of these linear maps:

1. Find $\mathrm{Ker}(p)$ and $\mathrm{Im}(p)$.
2. Draw them on the plane and check that $\mathrm{Ker}(p) \oplus \mathrm{Im}(p) = \mathbb{R}^2$.
3. Let $u \in E$, $v \in \mathrm{Im}(p)$ and $w \in \mathrm{Ker}(p)$ such that $u = v + w$.
   Calculate $p(v)$. Deduce $p(u)$ as a function of v and w.
   Propose a graphical construction of $p(u)$.

4. Explain why this linear map is called a "projector".
5. Check that $p \circ p = p$
6. Let $f = 2p - \mathrm{id}$. Study the image by f of an arbitrary vector from $\mathbb{R}^2$.
   How would you call this linear map?

Exercise 4.15
Let $f \in \mathscr{L}(\mathbb{R}^2)$ be defined by:

$$f : \begin{cases} \mathbb{R}^2 & \longrightarrow \mathbb{R}^2 \\ (x, y) & \longmapsto (4x - 6y, 2x - 3y) \end{cases}$$

1. Show that f is a projector, that is, $f \circ f = f$.
2. Determine $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ as spanned linear subspaces (using the Span notation), draw them on the plane.
3. Justify that $\mathrm{Ker}(f) \oplus \mathrm{Im}(f) = \mathbb{R}^2$.
4. Show that $\forall v \in \mathrm{Im}(f), \quad f(v) = v$.
5. Let $w \in \mathbb{R}^2$ and $(u, v) \in \mathrm{Ker}(f) \times \mathrm{Im}(f)$ such that $w = u + v$. What is $f(w)$?

Exercise 4.16
(Personal work) Describe the following projectors:

$$p_1 : \begin{cases} \mathbb{R}^3 & \longrightarrow \mathbb{R}^3 \\ (x, y, z) & \longmapsto \frac{1}{3}(x + y + z, x + y + z, x + y + z) \end{cases}$$

$$p_2 : \begin{cases} \mathbb{R}^3 & \longrightarrow \mathbb{R}^3 \\ (x, y, z) & \longmapsto (\frac{1}{2}(x - y), \frac{1}{2}(y - x), z) \end{cases}$$